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Generic Bifurcations of Second Order Ordinary Differential Equations near Closed Orbits

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1. INTRODUCTION

Much interest has been devoted to the generic bifurcations of vector fields near critical points and closed orbits. Well known are results of Sotomayor for one-parameter families of vector fields (see [14, 15]). Recently Dumortier [5] proved some genericity theorems for n-parameter families of vector fields on 2-dimensional manifolds and gave an explicit clasification for $n \leqslant 4$.

Since many physical phenomena are described by second order ordinary differential equations (ODE), it is useful to study their generic properties in such sense, that only perturbations satisfying the structure of second order equation are allowed. Generic properties of second order ODE were first studied by Shahshahani [12], who proved the analogous theorem to the theorem of Kupka and Smale for vector fields (see [1, 11, 13]).

The purpose of this paper is to study generic bifurcations of one-parameter families of second order ODE near closed orbits. The main result attained here is similar to the results of Sotomayor [14] for vector fields (see also [9], the similar problem for one-parameter families of diffeomorphisms was solved by Brunovský [3, 4]). Generic bifurcations of one-parameter families of second order ODE near critical points are studied in [10] and the results attained here are similar to the results attained for one-parameter families of vector fields (see [2, 8, 14]).

2. PRELIMINARIES

We suppose that A is a compact 1-dimensional C^r manifold and X is a compact n-dimensional C^{r+1} manifold. Let $T(X)$ denotes the tangent bundle of X. Denote by $\Gamma_I{}^r(TX)$ the set of C^r vector fields on $T(X)$ endowed with the Whitney C^r topology.

Let τ_X: $T(X) \to X$ be the natural projection. A vector field $\xi \in \Gamma_I{}^r(TX)$ is called a second order ODE on X if $D\tau_X \circ \xi = 1_{T(X)}$, where $D\tau_X$ denotes the

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differential of the map τ_X and $1_{T(X)}$ is the identical map of $T(X)$ onto $T(X)$. We will denote the set of such equations on X by $\Gamma^r_{II}(TX)$.

Denote by $H_I{}^r(A, TX)$ the set of all parametrized C^r vector fields on $T(X)$ (see [1, Sect. 21]). We suppose the Whitney C^r topology on $H_I{}^r(A, TX)$.

A parametrized vector field $\xi \in H_I{}^r(A, TX)$ is called a C^r parametrized second order ODE on X if $\xi_a \in \Gamma^r_{II}(TX)$ for all $a \in A$, where ξ_a is given by $\xi_a(x) = \xi(a, x)$ for $x \in T(X)$. We will denote the set of such equations by $H^r(A, X)$. This set is a closed subspace of $H_I{}^r(A, TX)$. If we endow this set with the topology induced by the topology on $H_I{}^r(A, TX)$, then $H^r(A, X)$ will have the Baire property, i.e., a countable intersection of open dense sets is dense.

A property P of a parametrized second order ODE is called generic in $H^r(A, X)$ if the set $\{\xi \in H^r(A, X) \mid P\}$ contains a residual set, i.e., a set which is a countable intersection of open and dense sets in $H^r(A, X)$.

The definition of the parametrized second order ODE implies that in local charts (U, α), (V, β) on A and X, respectively, the local representative ξ' of $\xi \in H^r(A, X)$ has the form

$$\xi'(\mu, x, v) = (x, v, v, \xi_{\alpha\beta}(\mu, x, v)), \tag{1}$$

where $\mu \in \alpha(U)$, $(x, v) \in \beta(V) \times R^n$, $\xi_{\alpha\beta}\colon \alpha(U) \times \beta(V) \times R^n \to R^n$ is a C^r map.

Let $\xi \in H^r(A, X)$, $a \in A$ and let γ be a closed orbit of the vector field ξ_a through x of a prime period τ. Then γ is called Δ-transversal if $\phi(\xi) \pitchfork_{(a,x,\tau)} \Delta$ ($\phi(\xi)$ transversally intersects Δ at (a, x, τ)), where $\Delta = \{(x, t, x) \mid x \in T(X),$ $t \in R^+ = (0, \infty)\}$, $\phi\colon H^r(A, X) \to C^r(A \times T(X) \times R^+, T(X) \times R^+ \times T(X))$ is given by $\phi(\xi) = \phi_\xi$ for $\xi \in H^r(A, X)$, $\phi_\xi(a, x, t) = (x, t, \varphi^\xi(a, x, t))$ for $(a, x, t) \in A \times T(X) \times R^+$, φ^ξ is the parametrized flow of ξ. By $H_\Delta{}^r(A, X)$ we denote the set of all $\xi \in H^r(A, X)$ such that if $a \in A$, then all closed orbits of the vector field are Δ-transversal.

Let $T(X) = \bigcup_{i=1}^\infty K_i$, where K_i $(i = 1, 2, 3,...)$ are compact subsets of $T(X)$ such that $K_i \subset K_{i+1}$ for all i. For a positive number L denote by $H_L{}^r(A, K_i)$ the set of $\xi \in H^r(A, X)$ such that for arbitrary (a, x_1), $(a, x_2) \in A \times K_i$, $\rho_2(\xi(a, x_1), \xi(a, x_2)) \leqslant L_1\rho_1(x_1, x_2)$, where $L_1 < L$, ρ_1, ρ_2 are metrics on $T(X)$ and $T^2(X)$, respectively, compatible with the topology on $T(X)$ and $T^2(X)$, respectively. By [6, Proposition 5.8] such metrics exist. It is clear that the set $H_L{}^r(A, K_i)$ is open.

LEMMA 2.1 (see [7, Theorem 4]). *Let $\xi \in H_L{}^r(A, K_i)$, $a \in A$, then arbitrary closed orbit of the vector field ξ_a contained in K_i has a prime period $\geqslant 4/L$.*

Let $p\colon A \times T(X) \times R^+ \to A \times T(X)$ be the projection and $Z \subset A \times T(X) \times R^+$. Denote $B(Z, \sigma) = \{(a, x, t) \in A \times T(X) \times R^+ \mid d(Z, (a, x, t)) < \sigma\}$, where $\sigma > 0$, $d(Z, (a, x, t)) = \inf_{z \in Z} r(z, (a, x, t))$, r is a metric on $A \times T(X) \times R^+$ compatible with the topology on $A \times T(X) \times R^+$. Let $B_p(Z, \sigma) = p(B(Z, \sigma))$

and $N(Z, \sigma) = A \times T(X) - \overline{B_p(Z, \sigma)}$. Denote $b_l = 4/L_l$, where $\{L_l\}_{l=1}^{\infty}$ is an increasing sequence of positive numbers such that $\lim_{l\to\infty} L_l = +\infty$. If $\xi \in H^r_{L_l}(A, K_i)$, $a \in A$, then by Lemma 2.1 all closed orbits of ξ_a contained in K_i have prime period $\geqslant b_l$.

For $\xi \in H^r(A, X)$ denote $Y_0(\xi) = \{(a, x) \in A \times T(X) \mid \xi(a, x) = 0_x\}$, where 0_x is the zero in $T_x(TX)$. Let $\{\delta_l\}_{l=1}$ be a sequence of positive numbers such that $\delta_l < \frac{1}{2} b_l$. For $\xi \in H^r(A, X)$ and q a natural number, define the map:

$$\phi_{jql}(\xi)\colon N\left(\bigcup_{s=0}^{j} Y_{sql}(\xi), q^{-1}\right) \times R^+ \to T(X) \times R^+ \times T(X),$$

$$\phi_{jql}(\xi) = \phi(\xi)/N\left(\bigcup_{s=0}^{j} Y_{sql}(\xi), q^{-1}\right) \times R^+, \qquad \text{where}$$

$$Y_{sql}(\xi) = p\{[\phi_{s-1,q,l}(\xi)]^{-1}(\Delta) \cap A \times T(X) \times (0, (s+1)b_l), \quad s = 1, 2,..., j,$$

$$Y_{0ql}(\xi) = Y_0(\xi) \times (0, b_l)$$

and $\phi(\xi)$ is the map defined before.

Now, define the sets:

$$H^r_{ljq}(K_i) = \{\xi \in H^r_{L_l}(A, K_i) \mid \phi_{jql}(\xi) \pitchfork \Delta \quad \text{on the set}$$

$$N_{ijql} = (A \times K_i) \cap N\left(\bigcup_{s=0}^{j} Y_{sql}(\xi), 2q^{-1}\right) \times [jb_l - \delta_l, (j+1)b_l - \delta_l],$$

$i, j, q, l = 1, 2, 3, \dots$.

Analogously to the proof of [9, Lemma 4], using [1, Theorems 18.2, 19.1], it is possible to prove

LEMMA 2.2. *The set $H^r_{ljq}(K_i)$ is open and dense in $H^r_{L_l}(A, K_i)$.*

Let $\xi \in H^r(A, X)$, $a_0 \in A$, $x_0 \in T(X)$ and let γ be a closed orbit of ξ_{a_0} passing through x_0 of a prime period τ. Let $\Sigma \subset T(X)$ be a $(2n-1)$-dimensional transverse section to γ at x_0. Denote by $H = H[\xi, a_0, x_0, \gamma, V_1 \times (V_2 \cap \Sigma)]$: $V_1 \times (V_2 \cap \Sigma) \to \Sigma$ the parametrized Poincaré mapping, where $W = V_1 \times V_2$ is an open neighbourhood of $(a_0, x_0) \in A \times T(X)$ (see [9, p. 74]).

Let $\hat{H}$: $V_1 \times (V_2 \cap \Sigma) \to \Sigma \times \Sigma$ be given by $\hat{H}(a, x) = (x, H(a, x))$ for $(a, x) \in V_1 \times (V_2 \cap \Sigma)$ and let $\Delta(\Sigma) = \{(x, y) \in \Sigma \times \Sigma \mid x = y\}$, which is a closed submanifold of $\Sigma \times \Sigma$ of dimension $2n - 1$.

Using the transversality condition for $\phi_{jql}(\xi)$ $(\phi_{jql}(\xi) \pitchfork \Delta)$ it is possible to prove

LEMMA 2.3. *If $\xi \in H^r_{ljq}(K_i)$, $(a_0, x_0, \tau_0) \in \overline{N_{ijql}}$, then*

1. $\hat{H} \pitchfork_{(a_0,x_0,\tau_0)} \Delta(\Sigma)$

2. *If there exists a closed orbit of* ξ_{a_0} *passing through* x_0 *of a prime period* τ_0, *then* $H^{-1}(\Delta(\Sigma))$ *is a* 1*-dimensional* C^r *submanifold of* $V_1 \times (V_2 \cap \Sigma)$ *for* $V_1 \times V_2$ *a sufficiently small neighborhood of* (a_0, x_0).

3. Fundamental Approximation Lemma

LEMMA 3.1. *Let* $\xi \in H^r_{ljq}(K_i)$, $(a_0, x_0, \tau_0) \in \overline{N_{ijql}}$ *and let* γ_{a_0} *be a closed orbit of* ξ_{a_0} *of a prime period* τ_0. *Let* $V_1 \times V_2$ *be a open neighbourhood of* (a_0, x_0) *in* $A \times T(X)$ *such that the Poincaré mapping* $H = H[\xi, a_0, x_0, \gamma_{a_0}, V_1 \times (V_2 \cap \Sigma)]$ *is defined. Let* W_1 *be an open neighbourhood of* a_0 *in* A *such that* $\overline{W}_1 \subset V_1$ *and let* W_2 *be an open neighbourhood of* x_0 *in* $T(X)$ *such that* $\overline{W}_2 \subset V_2$. *Let* $H_1 = H/\overline{W}_1 \times (\overline{W}_2 \cap \Sigma)$. *Then there exists a neighbourhood* $U(H_1)$ *of* H_1 *in* $C^r(\overline{W}_1 \times (\overline{W}_2 \cap \Sigma), \Sigma)$ *such that*

1. *for every* $\tilde{H}_1 \in U(H_1)$ *there is a* $\tilde{\xi} \in H_l{}^r(A, TX)$ *such that* $\varphi^{\tilde{\xi}}(a, x, \tau(a, x)) = \tilde{H}_1(a, x)$ *for all* $(a, x) \in \overline{W}_1 \times (\overline{W}_2 \cap \Sigma)$, *where* $\varphi^{\tilde{\xi}}$ *is the parametrized flow of* $\tilde{\xi}$, τ: $\overline{W}_1 \times \overline{W}_2 \to R$ *is a* C^r *function. Moreover,* $\tilde{\xi}$ *depends continuously on* $\tilde{H}_1$ *and if* $U(H_1)$ *is a sufficiently small neighbourhood of* H_1, *then for every* $a \in W_1$ *there exists a unique* $\tilde{x} \in \overline{W}_2 \cap \Sigma$ *such that* $\tilde{H}_1(a, \tilde{x}) = \tilde{x}$, *i.e., for every* $a \in W_1$ *the vector field* $\tilde{\xi}_a$ *has a unique closed orbit* $\tilde{\gamma}_a$ *in a sufficiently small neighbourhood of* γ_{a_0} *with prime period in the interval* $[jb_l - \delta_l, (j+1)b_l - \delta_l]$

2. *For* $U(H_1)$ *sufficiently small and for* $\tilde{H}_1 \in U(H_1)$ *and* $\tilde{\xi}$ *as in* 1, *there exists an* $\eta = \eta(\tilde{\xi}) \in H^r_{ljq}(K_i)$ *such that for* $a \in W_1$ *the vector field* η_a *is differentiably conjugate to* $\tilde{\xi}_a$ *near* $\tilde{\gamma}_a$.

Proof. Part 1 follows from [9, Lemma 6], Lemma 2.3 and from the openness of the transversality property.

To prove 2 suppose that the C^r map $\hat{\Theta}$: $W_1' \times \hat{B}^{2n-1} \times [-2, 2] \to \hat{F}$ is a parametrized flow-box of $\tilde{\xi}$, where W_1' is an open neighbourhood of a_0, $\hat{B}^{2n-1}$ is a closed disc with center at $0 \in R^{2n-1}$, $\hat{\Theta}_a$: $\hat{B}^{2n-1} \times [-2, 2] \to \hat{F}$ given by $\hat{\Theta}_a(x, t) = \hat{\Theta}(a, x, t)$ is a flow-box of $\tilde{\xi}_a$ for every $a \in W_1'$, i.e., $\hat{\Theta}_a$ is a C^r diffeomorphism, $\hat{\Theta}_a(\{b\} \times [-2, 2])$ is a trajectory of $\tilde{\xi}_a$ for $b \in \hat{B}^{2n-1}$ and $\hat{\Theta}_{a_0}(\{0\} \times [-2, 2]) = \gamma_{a_0} \cap \hat{F}$.

Let W_1'' be an open neighbourhood of a_0 such that $\overline{W_1'} \subset W_1'' \subset W_1$ and let B^{2n-1} be a closed disc in R^{2n-1} with center at 0 such that $\overline{\hat{B}^{2n-1}} \subset B^{2n-1}$. Denote $F = \hat{\Theta}(W_1'' \times B^{2n-1} \times [-1, 1])$.

Let $\tilde{\xi}$ be as in the assumptions and such that it is agreeing with ξ outside $\hat{F}$. We shall construct a parametrized second order ODE $\eta \in H^r(A, X)$ sufficiently close to $\tilde{\xi}$ agreeing with ξ outside $\hat{F}$ and such that for every $a \in W_1'$, η_a is differentiably conjugate to $\tilde{\xi}_a$ near γ_{a_0}. To constract such η, parametrize the trajectory of ξ_a in $\hat{F}$ as $\lambda_\xi^{x,a}(t) = (\lambda_{1,\xi}^{x,a}(t),\dots, \lambda_{2n,\xi}^{x,a}(t)) = \hat{\Theta}(a, x, t)$, $(a, x, t) \in W_1'' \times \hat{B}^{2n-1} \times [-2, 2]$. Since ξ_a is a second order ODE,

$$\frac{d}{dt}\lambda_{i,\xi}^{x,a}(t) = \lambda_{i+n,\xi}^{x,a}(t), \qquad i = 1, 2,\dots, n,\ t \in [-2, 2]$$

and hence

$$\lambda_{i,\xi}^{x,a}(+1) = \lambda_{i,\xi}^{x,a}(-1) + \int_{-1}^{+1} \lambda_{n+i,\xi}^{x,a}(s)\, ds, \qquad i = 1, 2,\dots, n.$$

Denote by $\lambda_{\tilde{\xi}}^{x,a}(t) = (\lambda_{1,\tilde{\xi}}^{x,a}(t),\dots, \lambda_{2n,\tilde{\xi}}^{x,a}(t))$ the trajectory of $\tilde{\xi}_a$ in $\hat{F}$ such that

$$\lambda_{i,\tilde{\xi}}^{x,a}(-1) = \lambda_{i,\xi}^{x,a}(-1), \qquad i = 1, 2,\dots, n.$$

To construct η, we seek a C^r mapping

$$\gamma^x\colon W_1'' \times [-2, 2] \to \hat{F}$$

such that for every $a \in W_1''$ the curve $\gamma_a{}^x\colon [-2, 2] \to \hat{F}$ given by $\gamma_a{}^x(t) = \gamma^x(a, t)$ has the following properties:

$$\gamma^{x,a}(\pm 1) = \lambda_{\tilde{\xi}}^{x,a}(\pm 1) = \tilde{P}_x{}^{\pm}(a) = (\tilde{P}_{1,x}^{\pm}(a),\dots, \tilde{P}_{2n,x}^{\pm}(a)) \tag{2}$$

(Note that $\lambda_{\tilde{\xi}}^{x,a}(-1) = P_x{}^-(a) = \tilde{P}_x{}^-(a)$),

$$\left.\frac{d^j\gamma_i^{x,a}(t)}{dt^j}\right|_{t=\pm 1} = \left.\frac{d^j\lambda_{i,\tilde{\xi}}^{x,a}(t)}{dt^j}\right|_{t=\pm 1}, \qquad i = 1, 2,\dots, 2n,\ j \leqslant r \tag{3}$$

Consider a family of functions

$$\begin{aligned} \phi_t(s) &= 1 + t \exp\frac{s^2}{s^2 - 1} && \text{for}\quad s \in [-1, 1] \\ &= 0 && \text{elsewhere}, \end{aligned}$$
$$t \in [-2, 2].$$

Define the mappings:

$$f_{i,\tilde{\xi}}^x : W_1'' \times [-2, 2] \to R, \qquad i = 1, 2,\dots, n,$$

$$(a, t) \to \int_{-1}^{1} \phi_t(s)\, g_i(a, \lambda_{\tilde{\xi}}^{x,a}(s))\, ds + P_{i,x}^-(a),$$

where $g = (g_1, g_2,\dots, g_{2n})$ is the local representative of $\tilde{\xi}$ in $\hat{F}$ and similarly

$$f_{i,\xi}^x : W_1'' \times [-2, 2] \to R, \qquad i = 1, 2,\dots, n$$

$$(a, t) \to \int_{-1}^{1} \phi_t(s)\, \lambda_{n+i,\xi}^{x,a}(s)\, ds + P_{i,x}^-(a).$$

Define $f_{i,\xi}^{x,a}(t) = f_{i,\xi}^x(a, t)$ and $f_{i,\tilde{\xi}}^{x,a}(t) = f_{i,\tilde{\xi}}^x(a, t)$ for $a \in W_1''$ fixed and $t \in [-2, 2]$.

We may assume that $\lambda_{n+i,\xi}^{x,a}(t) \neq 0$ for $i = 1, 2,\ldots, n$ and for all $t \in [-1, 1]$, since if not then we can perturbe the trajectories as follows

$$\begin{aligned} \tilde{\lambda}_i^{x,a}(t) &= P_i^- + \epsilon \int_{-1}^{t} \exp \frac{s^2}{s^2 - 1}\, ds \qquad \text{for} \quad t \in [-1, 1] \\ &= P_i^- \qquad \text{elsewhere,} \\ \tilde{\lambda}_{i+n}^{x,a}(t) &= \epsilon \exp \frac{t^2}{t^2 - 1} \qquad \text{for} \quad t \in [-1, 1] \\ &= 0 \qquad \text{elsewhere,} \end{aligned}$$

where $\epsilon > 0$, which define a second order ODE sufficiently close to ξ if ϵ is sufficiently small. Thus suppose that $\lambda_{n+i,\xi}^{x,a}(t) \neq 0$ for $i = 1, 2,\ldots, n$, $t \in [-1, 1]$. Since ξ is a second order ODE

$$f_{i,\xi}^{x,a}(0) = \int_{-1}^{+1} \lambda_{n+i,\xi}^{x,a}(s)\, ds + P_{i,x}^-(a) = \lambda_{i,\xi}^{x,a}(+1) = P_{i,x}^+(a)$$

for $i = 1, 2,\ldots, n$.

Since $(d/dt) f_{i,\xi}^{x,a}(t) = \int_{-1}^{+1} \exp(s^2/(s^2 - 1))\, \lambda_{n+i,\xi}^{x,a}(s)\, ds$ and $\lambda_{n+i,\xi}^{x,a}(s) \neq 0$ for all $s \in [-1, 1]$, the functions $f_{i,\xi}^{x,a}(t)$, $i = 1, 2,\ldots, n$ are strictly monotone on the interval $[-2, 2]$. Therefore for $\tilde{\xi}$ sufficiently close to ξ there exists a $t_i = t_i(a, \tilde{\xi}) \in (-1, 1)$ such that $P_{i,x}(a) = f_{i,\xi}^{x,a}(t_i(a, \tilde{\xi}))$ i.e., $P_{i,x}^+(a) = P_{i,x}^-(a) + \int_{-1}^{+1} \phi_{t_i(a,\tilde{\xi})}(s)\, \lambda_{n+i,\xi}^{x\ a}(s)\, ds$. Moreover, $t_i = t_i(a, \tilde{\xi})$ depends continuously on a and $\tilde{\xi}$. Define $\gamma_i^{x,a}(t) = P_{i,x}^-(a) + \int_{-1}^{t} \phi_{t_i(\tilde{\xi})}(s)\, \lambda_{n+i,\xi}^{x,a}(s)\, ds$, $\gamma_{i+n}^{x,a}(t) = \phi_{t_i(\tilde{\xi})}(t)\, \lambda_{n+i,\xi}^{x,a}(t)$ for $i = 1, 2,\ldots, n$, $(a, x, t) \in W_1'' \times \hat{B}^{2n-1} \times [-2, 2]$. From the construction it is clear that the mapping γ: $W_1'' \times \hat{B}^{2n-1} \times [-2, 2] \to \hat{F}$, $(a, x, t) \to (\gamma_1^{x,a}(t),\ldots, \gamma_{2n}^{x,a}(t))$ is C^r. If $\tilde{\xi}$ is sufficiently close to ξ and W_1'' is sufficiently small, then for every a W_1'' the mapping

$$h^a\colon (\lambda_{1,\xi}^{x,a}(t),\ldots, \lambda_{2n,\xi}^{x,a}(t)) \to (\gamma_1^{x,a}(t),\ldots, \gamma_{2n}^{x,a}(t))$$

is a C^r diffeomorphism of $\hat{F}$ onto $h^a(\hat{F})$. Moreover,

$$E = \overline{\bigcup_{a \in W_1''} h^a(F)} \subset \hat{F}.$$

Let Ψ: $A \times T(X) \to R$ be a C^r function such that

$$\begin{aligned} \psi(a, z) &= 1 \qquad \text{on} \quad W_1' \times E \\ &= 0 \qquad \text{outside} \quad W_1'' \times \hat{F}. \end{aligned}$$

Define the parametrized vector field η as follows:

$$\begin{aligned} \eta(a, z) &= \xi(a, z) + \psi(a, z)[h_*^a \tilde{\xi}_a(z) - \xi(a, z)] \qquad \text{for} \quad (a, z) \in W_1'' \times \hat{F} \\ &= \xi(a, z) \qquad \text{outside} \quad W_1'' \times \hat{F}, \end{aligned}$$

where $h^a_* \tilde{\xi}_a(z) = Th^a \circ \tilde{\xi}_a \circ h^a(z)$. This vector field is a C^r parametrized second order ODE such that for every $a \in W_1$ the vector field η_a is differentiably conjugate to $\tilde{\xi}_a$ near γ_a, where γ_a is a closed orbit of $\tilde{\xi}_a$ sufficiently close to γ_{a_0}.

4. Quasi-hyperbolic Closed Orbits

Let $\xi \in H_I^r(A, TX)$ and let γ be a closed orbit of ξ_{a_0} passing through x_0. For $H = H[\xi, a_0, x_0, \gamma, V_1 \times (V_2 \cap \Sigma)]$ define the sets

$Z_k(H) = \{(a, x) \in V_1 \times (V_2 \cap \Sigma) \mid H_a^k(x) = x,\ H_a^j(x) \neq x \text{ for } 0 < j < k\}$, $k = 1, 2, \ldots$, where $H_a^1(x) = H_a(x) = H(a, x)$, $H_a^k(x) = H_a(H_a^{k-1}(x))$.

Denote by $P_i(\xi)$ $(i = 1, 2, 3)$ the set of $(a, x) \in V_1 \times (V_2 \cap \Sigma)$ such that the vector field ξ_a has a closed orbit passing through x of a prime period τ and

(a) $(a, x) \in P_1(\xi)$ if $\lambda = 1$ is the eigenvalue of the mapping $T_x\varphi_{(\tau,a)}$ of multiplicity 2;

(b) $(a, x) \in P_2(\xi)$ if $\lambda = -1$ is the eigenvalue of the mapping $T_x\varphi_{(\tau,a)}$ of multiplicity 1;

(c) $(a, x) \in P_3(\xi)$ if the mapping $T_x\varphi_{(\tau,a)}$ has a complex eigenvalue λ with $|\lambda| = 1$ and there are no other complex eigenvalues of this map with the absolute value equal to 1 except of complex conjugate $\bar{\lambda}$. ($\varphi_{(\tau,a)}: T(X) \to T(X)$, $\varphi_{(\tau,a)}(x) = \varphi(a, x, \tau)$, where φ is the parametrized flow of ξ).

By [9, 14] (see also Brunovský [4]), there exists an open, dense subset $G^r_{lјqk}(K_i) \subset H_I^r(A, TX)$ such that if $\xi \in G^r_{ljqk}(K_i)$, $(a_0, x_0, \tau) \in (A \times K_i) \cap N(\bigcup_{s=0}^{j} Y_s(\xi), 2q^{-1}) \times [jb_l - \delta_l, (j+1)b_l - \delta_l]$ and there is a closed orbit γ of ξ_{a_0} passing through x_0 of a prime period τ, then for the corresponding Poincaré mapping $H = H[\xi, a_0, x_0, \gamma, V_1 \times (V_2 \cap \Sigma)]$ the following is true

(1) The set $P_i(\xi)$ $(i = 1, 2, 3)$ is either empty or consists of isolated points

(2) *If* $(a_0, x_0) \in P_1(\xi)$, then near (a_0, x_0) the map H is locally of the form

$$\begin{aligned} y_2 &= y_1 + \alpha_1\mu + \alpha_2 y_1^2 + \omega(\mu, y, z), \\ z_2 &= Bz_1 + X(\mu, y_1, z_1), \end{aligned} \tag{4}$$

where $\dim y_1 = 1$, $\dim z_1 = 2n - 2$, ω, $X \in C^r$, $X(0, 0, 0) = 0$, $\omega(\mu, y_1, 0)$ contains only μ^2, μy_1 and terms of higher order than 2, $\alpha_2 \neq 0$ and B is a matrix which has no eigenvalue on the unit sphere and has no complex eigenvalue λ such that $\lambda^k = 1$

(3) If $(a_0, x_0) \in P_2(\xi)$, then near (a_0, x_0) the map H is of the form

$$\begin{aligned} y_2 &= -y_1 + \alpha_1\mu y_1 + \alpha_2 y_1^2 + \gamma_1 y_1^3 + \omega(\mu, y_1, z_1), \\ z_2 &= Bz_1 + X(\mu, y_1, z_1), \end{aligned} \tag{5}$$

where ω, X and B are as in (1) and $\alpha_2^2 + \gamma_1 \neq 0$

(4) If $(a_0, x_0) \in P_3(\xi)$, then near (a_0, x_0) the map H is of the form

$$\begin{aligned} u_2 &= \lambda(\mu)u_1 + \beta(\mu) \mid u_1 \mid^2 u_1 + X_1(\mu, y_1, u_1, u_1), \\ y_2 &= B(\mu)y_1 + X(\mu, y_1, u_1, u_1), \end{aligned} \tag{6}$$

where u_1 is complex, $\dim y_1 = 2n - 3$, $\mu \to \lambda(\mu)$, $\mu \to B(\mu)$ are of class C^{r-1}, $X_1 = o(\mid u_1 \mid^3 + \mid y_1 \mid)$, $X = o(\mid \mu_1 \mid + \mid y_1 \mid)$ for small μ and $\operatorname{Re} \beta(\mu)/\lambda(\mu) \neq 0$ for small μ.

We say that x_0 is a quasi-hyperbolic fixed point (cf. Sotomayor [14]) of the Poincaré mapping H_{a_0} of the vector field ξ_{a_0} $(H = H[\xi, a_0, x_0, \gamma, V_1 \times (V_2 \cap \Sigma))$ if and only if for some central manifold W^c, H_{a_0}/W^c is C^r conjugate to one of the following C^r diffeomorphisms:

(a) $y_2 = y_1 + \alpha_2 y_1^2 + o(y_1^2)$, $\alpha_2 \neq 0$

(b) $y_2 = -y_1 + \alpha_2 y_1^2 + \gamma_1 y_1^3 + o(\mid y_1 \mid^3)$, $\gamma_1 + \alpha_2^2 \neq 0$

(c) $u_2 = \lambda u_1 + \beta \mid u_1 \mid^2 u_1 + o(\mid u_1 \mid^2)$,

u_1 is complex, λ, λ^2, λ^3, $\lambda^4 \neq 1$, $\operatorname{Re} \beta/\lambda \neq 0$. The corresponding closed orbit orbit of ξ_{a_0} passing through x_0 is called quasi-hyperbolic.

It is obvious that if $(a_0, x_0) \in \bigcup_{i=1}^3 P_i(\xi)$, then the point x_0 is a quasi-hyperbolic fixed point of H_{a_0} and the closed orbit passing through x_0 is quasi-hyperbolic.

Using Lemma 3.1 and the properties of the set $G^r_{ljq}(K_i)$, it is possible to prove the following lemma.

LEMMA 4.1. *There is an open, dense subset $\tilde{H}^r_{ljq}(K_i)$ of $H^r_{ljq}(K_i)$ such that if*

$$\xi \in \tilde{H}^r_{ljq}(K_i),\ (a_0, x_0, \tau_0) \in (A \times K_i) \cap \overline{N\left(\bigcup_{s=0}^{j} Y_s(\xi), 2q^{-1}\right)} \times [jb_l - \delta_l, (j+1)b_l - \delta_l]$$

and γ is a closed orbit of ξ_{a_0} of a prime period τ_0, then there exists a $\tilde{\xi} \in G^r_{ljq}(K_i)$ sufficiently close to ξ (in the topology on $H_l{}^r(A, TX)$) such that for $a \in V_1$ (V_1 is a sufficiently small neighbourhood of a_0) $\tilde{\xi}_a$ is differentiably conjugate to ξ_a on some neighbourhood of γ, i.e., if $(a_0, x_0) \in P_1(\xi)$ $(P_2(\xi), P_3(\xi))$, then the corresponding Poincaré mapping $H = H[\xi, a_0, x_0, \gamma, V_1 \times (V_2 \cap \Sigma)]$ is locally of the form (4) [(5), (6)].

5. THE THEOREM

THEOREM. *There exists a residual set $H_1{}^r(A, X) \subset H^r(A, X)$ such that if $\xi \in H_1{}^r(A, X)$, then for all $a \in A$ the second order* ODE *ξ_a has only hyperbolic and quasi-hyperbolic closed orbits and*

(1) *If γ_{a_0} is a quasi-hyperbolic closed orbit of ξ_{a_0}, then there is an open neighbourhood V of a_0 such that for all $a \in V$ ξ_a has no quasi-hyperbolic closed orbit in a neighbourhood of γ_{a_0}.*

(2) *If γ_{a_0} is a closed orbit of ξ_{a_0} passing through x_0, then there is a chart $(V_1 \times V_2, h_1 \times h_2)$ on $A \times T(X)$ at (a_0, x_0), $h_1(a_0) = 0$, $h_2(x_0) = 0$ such that if $H = H[\xi, a_0, x_0, \gamma_{a_0}, V_1 \times (V_2 \cap \Sigma)]$ is the corresponding Poincaré mapping, then*

(a) *$Z_1 = Z_1(H)$ is an 1-dimensional C^r submanifold of $V_1 \times (V_2 \cap \Sigma)$*

(b) *If $(a_0, x_0) \in P_1(\xi)$, then $(h_1 \times h_2)(Z_1(H)) = \{(\mu, y_1, \ldots, y_{2n-1}) \mid \mu = \varphi_0(y_1), y_i = \varphi_i(y_1), i = 1, 2, \ldots, 2n - 1, y_1 \in J\}$, where J is an open interval, $0 \in J$, $\varphi_i \in C^r$, $i = 1, 2, \ldots, 2n - 1$, $\varphi_0(0) = 0$, $d\varphi_0(0)/dy_1 = 0$, $d^2\varphi_0(0)/dy_1^2 > 0$ and for $\mu > 0$ ξ_a $(a = h_1^{-1}(\mu))$ has exactly two closed orbits γ_1, γ_2 in a neighbourhood of γ_{a_0}, which are hyperbolic. The index of γ_1 is equal to the index of γ_{a_0} and the index of γ_2 is equal to the index of γ_{a_0} minus 1. Moreover, $V_1 \times (V_2 \cap \Sigma) - Z_1(H)$ contains no invariant set.*

(c) *If $(a_0, x_0) \in P_2(\xi)$, then $\bar{Z}_2 = \overline{Z_2(H)}$ is an 1-dimensional C^{r-1} submanifold of $V_1 \times (V_2 \cap \Sigma)$, $\bar{Z}_2 - Z_2 = P_2(\xi)$ and $V_1 \times (V_2 \cap \Sigma) - (Z_1 \cup Z_2)$ contains no invariant set. This means that if $\mu = h_1(a)$, $\mu_0 = h_1(a_0)$ and τ_0 is a prime period of γ_{a_0}, then for any μ close to μ_0 there is one closed orbit of ξ_a the period of which tends to τ_0 as $\mu \to \mu_0$ and there is another closed orbit of ξ_a for the value of μ on one side of μ_0, the period of which tends to $2\tau_0$ as $\mu \to \mu_0$.*

(d) *If $(a_0, x_0) \in P_c(\xi)$, then H_{a_0} has the composed focus fixed point, which give rise to a 2-dimensional torus as the saturated set of the closed orbit γ_a.*

Proof. Define the sets $H_{lqk}(K_i) = \bigcap_{j=1}^{q} H_{ljk}(K_i)$, $K_{qk} = \bigcup_{i,l=1}^{\infty} H_{ljk}(K_i)$. The set K_{qk} is open in $H^r(A, X)$. Since $H^r(A, X) = \bigcup_{i,l=1}^{\infty} H_{L_l}^r(A, K_i)$, $K_{qk} \supset \bigcup_{i,l=1}^{\infty} K_{ljk}(K_i) = \bigcup_{i,l=1}^{\infty} H_{L_l}^r(A, K_i) = H^r(A, X)$, i.e., the set K_{qk} is dense in $H^r(A, X)$. Therefore the set $H_1{}^r(A, X) = \bigcap_{k,l=1}^{\infty} K_{qk}$ is residual in $H^r(A, X)$. The properties of the sets K_{qk} implies that $H_1{}^r(A, X)$ has the properties (1), (2) of our theorem.

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